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Topological Quantum Fields Theory

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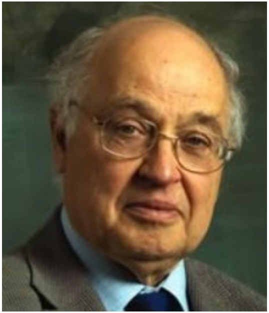
Outline

1. Preliminaries
2. TQFT
3. Formal properties of TQFT

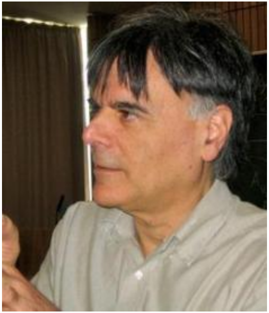
Preliminaries

Historical Background

- We have arrived at a description of a field theory as a functor from a **cobordism category** to the category of **Hilbert spaces**
- Originally it was **Segal** who first promoted this idea in **1989**, in the case of **conformal field theory**
- It was **Atiyah** who decided to reformulate Segals viewpoint in terms of a precise set of axioms for a **topological field theory**
- When one says the words **topological quantum field theory** to an academic audience, various things.
- Most mathematicians will probably understand the term to refer to the **Atiyah definitions**.
- Physicists, on the other hand, are probably more familiar with an alternative working definition of a topological field theory.



Michael Atiyah



Graeme Segal

Edward Witten

TQFT

- A **topological quantum field theory** (or topological field theory or TQFT) is a quantum field theory which *computes topological invariants.*
- Although TQFTs were invented by physicists, they are also of mathematical interest, being related to, among other things, **knot theory** and the **theory of four-manifolds** in algebraic topology, and to the theory of **moduli spaces** in algebraic geometry.
- Donaldson, Jones, Witten, and Kontsevich have all won **Fields Medals** for work related to topological field theory.
- The known topological field theories fall into two general classes: **Schwarz-type** TQFTs and **Witten-type** TQFTs. Witten TQFTs are also sometimes referred to as cohomological field theories.
- In Schwarz-type TQFTs, the correlation functions computed by the **path integral are topological invariants** because the path integral measure and the quantum field observables are explicitly independent of the metric.
- In Witten-type topological field theories, the topological invariance is more subtle. For example the Lagrangian for the WZW model does depend explicitly on the metric, but one shows by calculation that the expectation value of the partition function and a special class of

Quantization

- **Quantization** converts *classical fields* into operators acting on *quantum states of the field theory.*
- **Canonical quantization** of a field theory is analogous to the construction of *quantum mechanics* from *classical mechanics*.
- **Quantum field theory (QFT)** provides a theoretical framework for constructing quantum mechanical models of systems classically represented by an infinite number of degrees of freedom, that is, fields and (in a condensed matter context) many-body systems.
- It is the natural and quantitative language of *particle physics* and *condensed matter physics.*
- Most theories in modern particle physics, including the *Standard Model of elementary particles and their interactions*, are formulated as *relativistic quantum field theories.*
- Quantum field theories are used in many contexts, and are especially vital in *elementary particle physics*, where the particle count/number may change over the course of a reaction.

- **A conformal field theory (CFT)** is a quantum field theory also recognized as a *statistical mechanics model* at the critical point, that is invariant under conformal transformations i.e. transformations that preserve angles but not lengths.
- Conformal field theory has important applications in string theory, statistical mechanics, and condensed matter physics
- Conformal field theory has important applications in string theory, statistical mechanics, and condensed matter physics.
- The theory was first proposed by *Leigh Page* and *Norman I. Adams*.

Category Theory

What is a category?

A **Category** C Consist of some data

- A collection of **objects** $Ob(C)$,
- A Collection of **arrows** $Mor(C)$ between objects,
- A **Composition** $fog : a \to c$ for any two arrows $f : a \to b$ and $g : b \to c$.
- For any object a there is an identity arrow 1_a,

and some rules

- $fo(goh) = (fog)oh$
- Right unit law $1_b of = f$ and left unit law $go1_b = g$.

A **Functor** is a morphism between two Categories $F : C \to D$ s.t.

- F assign an object $F(a)$ in D for any object a in C.
- F assign an morphism $F(f) : F(a) \to F(b)$ in D for any morphism $f : a \to b$ in C.
- $F(1_a) = 1_{F(a)}$ and $F(fog) = F(f)oF(g)$

Monoidal Category

Let C be a category, then it is **monoidal category** if

- It has a **unit object** S,
- It has a **product** functor $\otimes : C \times C \to C$,
- The product is **unital** ($a \otimes S \simeq_{l(a)} a$, $S \otimes a \simeq_{r(a)} a$) and **associative** ($a \otimes (b \otimes c) \simeq_{\alpha_{a,b,c}} (a \otimes b) \otimes c$)
- Unital and associative natural transformations should be well defined

A monoidal category is **Braided** if there is a natural isomorphism $\beta_{A,B} : A \otimes B \to B \otimes A$ and

$$\beta_{A,B\otimes C} = (\beta_{A,C} \otimes Id_B) o (Id_A \otimes \beta_{B,C}) \quad \beta_{A\otimes B,C} = (Id_B \otimes \beta_{A,C}) o (\beta_{B,C} \otimes Id_C).$$

We call β a **Braiding** and a braided monoidal category is **symmetric** if the braiding be idempotent which means that

$$\beta_{A,B}\beta_{B,A} \simeq Id_{A\otimes B}$$

Functors

A Functor $F : C \to D$ between two symmetric monoidal categories is **Symmetric monoidal functor** if

- There exists a natural transformation $\phi : F(A) \otimes_D F(B) \to F(A \otimes_C B)$
- There is a morphism $\lambda : S_D \to F(S_C)$ between units of two **SMC**s.
- When ϕ, λ are i*dentity isomorphism*, or *relax*, we call F *strict*, *strong* or *lax* functor.

We usualy work with strong functors.
Also we can **define symmetric natural transformations** in a canonical way.

Duality in categories

We say that a SMC C has **duaity** if every object X is *dualizable* i.e. there is

- An object DX
- Morphisms $\eta : S \to X \otimes DX$
- Morphisms $\varepsilon : DX \otimes X \to S$

Such that the following composites are identity morphisms:

- $X \simeq S \otimes X \to X \otimes DX \otimes X \to X \otimes S \simeq X$
- $DX \simeq DX \otimes S \to DX \otimes X \otimes DX \to S \otimes DX \simeq DX$

$*-$categories

A $*-$category is a category C with a contravariant involution $*: C \to C$ which is identity on objects. We must have $X^* = X, (gf)^* = g^* f^*$

- The target for a TQFT is a symmetric monoidal $*-$category with duality.
- Morphisms $\eta : S \to X \otimes DX$
- Morphisms $\varepsilon : DX \otimes X \to S$

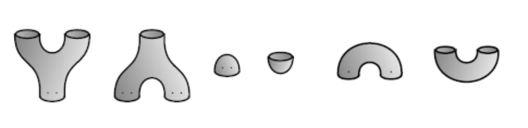
Such that the following composites are identity morphisms:

- $X \simeq S \otimes X \to X \otimes DX \otimes X \to X \otimes S \simeq X$
- $DX \simeq DX \otimes S \to DX \otimes X \otimes DX \to S \otimes DX \simeq DX$

What is Cobordism?

A rough of n-dimensional cobordism theory (**nCob**)

- **Objects** are oriented closed (that is, compact and without boundary) $n-1$ manifolds Σ.
- **Arrows** $M : \Sigma \to \Sigma'$ are compact oriented n-manifolds M which are cobordism from Σ to Σ'.
- **Composition** of cobordisms $M : \Sigma \to \Sigma'$ and $N : \Sigma' \to \Sigma''$ is defining by gluing M to N along Σ'.
-



Toplogical Quantum Fields Theory

TQFT

Definition of TQFT

A **TQFT** Z in dimension d over C is a SMC functor

$$Z : Cob(d) \to C.$$

In details it means that,

- For each $d-$manifold Σ we have an object $Z(\Sigma)$,
- For each diffeomorphism $f : \Sigma_0 \to \Sigma_1$ we have an isomorphism $Z(f) : Z(\Sigma_0) \to Z(\Sigma_1)$,
- For each $(d+1)-$manifold M with $\delta M = \Sigma_0^{op} \amalg \Sigma_1$ we have a morphism $Z(f) : Z(\Sigma_0) \to Z(\Sigma_1)$,
- Equivalent cobordisms gives the same morphism.

When $C = \mathbf{Vec}_k$ symmetric monoidal category of vector spaces it is a well known TQFT,

Definition of involutory TQFT

A **TQFT** without duality is **involutory** if there is an isomorphism

$$\delta : Z(\Sigma^{op}) \to Z(\Sigma)^*$$

Atyiah TQFT

- Atiyah suggested a set of axioms for topological quantum field theory which was inspired by **Segal's** proposed axioms for conformal field theory, and the **Witten**'s idea of the geometric meaning of supersymmetry.
- Atiyah's axioms are constructed on gluing the boundary with differentiable (topological or continuous) transformation, while Segal's are with conformal transformation.
- These axioms have been relatively **useful** for mathematical treatments of Schwarz-type QFTs, although **it isn't clear** that they capture the whole structure of Witten-type QFTs.
- The basic idea is that a TQFT is a **functor** from a certain category of *cobordisms* to the *category of vector spaces.*
- There are in fact *two different sets* of axioms which could reasonably be called the Atiyah axioms. These axioms differ *basically* in whether or not they *study a TQFT defined on a single fixed n-dimensional Riemannian / Lorentzian spacetime M* or a *TQFT defined on all n-dimensional spacetimes at once.*

Atyiah TQFT

These data are subject to the following axioms:

1. Z is functorial with respect to orientation preserving diffiomorphisms of Σ and M
2. Z is involutory i.e. $Z(\Sigma^*) = Z(\Sigma)^*$
3. Z is multiplicative, i.e $Z(M + M') = Z(M) \otimes Z(M')$.
   Furthermore, Atiyah adds two axiom to them. Namely, they are (4) and (5).
4. $Z(\emptyset) = \Gamma$ for the $d-$dimensional emty manifold and $Z(\emptyset) = 1$ for the $(d+1)-$dimensional empty manifold.
5. $Z(M^*) = \overline{Z(M)}$ (hermitian axiom).

Atiyah TQFT

A **TQFT** in the sense of Atiyah is the same as involutory TQFT without duality.

TQFT with duality

Atiyah TQFT

Let Z be an involutory **TQFT** in C s.t. $Z(f)$ is isometry for each diffeomorphism f. We say that Z is a TQFT with duality if for any cobordism $M : \Sigma_0 \to \Sigma$.

$$Z(M^{op}) = Ad(Z(M)) : Z(\Sigma_1) \to Z(\Sigma_0)$$

An Example d=0

In case of $d = 0$, Σ consist of finitly many points. and

- For any single point p let $V = Z(p)$
- For S set of n point let $Z(S) = V \otimes V \otimes ... \otimes V = V^{\otimes n}$.
- S_n acts on $V^{\otimes n}$
- A standard way to get the *quantum Hilbert space* is to give a classical symplectic manifold and then quantize it.

Topological QFT's with $d = 0$ relate naturally to the classical representation theory of *Lie groups* and *symmetric groups*.

Formal properties of TQFT

Frobenius Algebra

In mathematics, especially in the fields of representation theory and module theory, a **Frobenius algebra** is a *finite dimensional unital associative algebra* with a special kind of bilinear form which gives the algebras particularly nice duality theories.

Frobenius algebras began to be studied in the 1930s by *Brauer* and *Nesbitt* and were named after **Frobenius.**

Definition

A finite dimensional, unital, associative algebra A defined over a field k is said to be a **Frobenius algebra** if:

- A is equipped with a nondegenerate bilinear form $\sigma : A \times A \to k$
- $\sigma(a.b, c) = \sigma(a, b.c)$ *Frobenius form*
- Equivalently, one may equip A with a linear functional $\lambda : A \to k$ such that the kernel of λ contains no nonzero left ideal of A.

Examples

- Any matrix algebra over k with frobenius form $\sigma(a, b) = Tr(a.b)$
- For a field k, the four-dimensional $k-$algebra $k[x, y]/(x^2, y^2)$ is a Frobenius algebra.

Frobenius Algebra Properties

Properties

- The direct product and tensor product of Frobenius algebras are Frobenius algebras.
- A finite-dimensional commutative local algebra over a field is Frobenius if and only if the right regular module is injective, if and only if the algebra has a unique minimal ideal.
- The right regular representation of a Frobenius algebra is always injective.

Categorical Definition

In a monoidal category $(C, \otimes, I)$ A frobenius object is a bimonoid (monoid and comonoid) $(A, \mu, \eta, \delta, \varepsilon)$ which (A, μ, η) is a monid and $(A\delta, \varepsilon)$ is a comonoid where the following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\delta \otimes A} & A \otimes A \otimes A \\ \mu \downarrow & & \downarrow A \otimes \mu \\ A & \xrightarrow{\delta} & A \otimes A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{A \otimes \delta} & A \otimes A \otimes A \\ \mu \downarrow & & \downarrow \mu \otimes A \\ A & \xrightarrow{\delta} & A \otimes A \end{array}$$

Frobenius Algebra and TQFT

Frobenius Algebra and TQFT

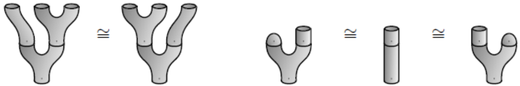
A commutative Frobenius algebra determines uniquely (up to isomorphism) a $(1+1)-$dimensional TQFT.
More precisely, the *category of commutative Frobenius K-algebras* is equivalent to the category of symmetric strong monoidal functors from $2 - Cob$ (the category of 2-dimensional cobordisms between 1-dimensional manifolds) to $\mathbf{Vect}_K$ (the category of vector spaces over K).

Sketch of Proof

- 1-dimensional manifolds are disjoint unions of circles: a TQFT associates a vector space with a circle, and the tensor product of vector spaces with a disjoint union of circles,
- A TQFT associates (functorially) to each cobordism between manifolds a map between vector spaces,
- The map associated with a pair of pants (a cobordism between 1 circle and 2 circles) gives a product map $V \otimes V \to V$ or a coproduct map $V \to V \otimes V$, depending on how the boundary components are grouped which is commutative or cocommutative,
- the map associated with a disk gives a counit (trace) or unit (scalars),

Frobenius Algebra and TQFT

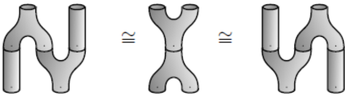
$$\mu : V \otimes V \to V, \quad \delta : V \to V \otimes V, \quad \eta : K \to V, \varepsilon : V \to K \quad dual \phi$$



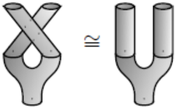
$$\mu(id \otimes \mu) \simeq \mu(\mu \otimes id) \qquad \mu(\delta \otimes id) = id = \mu(id \otimes \mu)$$

$$(\delta \otimes id)\delta \simeq (id \otimes \delta)\delta \qquad (\varepsilon \otimes id)\delta = id = (id \otimes \varepsilon)\delta$$

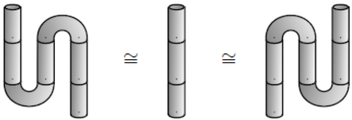
Frobenius Algebra and TQFT



$$(id \otimes \mu)(\delta \otimes id) \simeq \delta\mu \simeq (\mu \otimes id)(id \otimes \delta)$$



$$\mu(id_B \otimes id_A) \simeq \mu(id_A \otimes id_B), commutative$$



Nondegeneracy

Some aplications

Some aplications

- Seiberg-Witten gauge theory,
- topological string theory,
- relationship between knot theory and quantum theory,
- knot invariants

Thank You

Any Question?

References

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